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On the Novikov algebra structures adapted to the automorphism structure of a Lie group

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Abstract

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. The commutator of a Novikov algebra is a Lie algebra in which there exists a special affine structure (connection with zero curvature and torsion) defined by the Novikov algebra. For ensuring the consequences for the group structure, we need consider the more intrinsic connections defined by Novikov algebra structures, that is, the connections which are adapted to the automorphism structure of a Lie group. The resultant Novikov algebra is called a derivation algebra which satisfies every left multiplication operator is a derivation of its sub-adjacent Lie algebra. In this paper, we commence a study of the Novikov derivation algebras and as a consequence, we can construct Novikov algebras on some 2-solvable Lie algebras.

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1. Introduction

Poisson brackets of hydrodynamic type were introduced and studied in Refs. [1–4]

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \tag{1.1}$$

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The simplest local Lie algebra arising from brackets of hydrodynamic type (1.1) was also introduced as follows [3]:

$$g^{ij} = \sum_{k=1}^{n} C_k^{ij} u^k + g_0^{ij}, \quad b_k^{ij} = \text{const.}, \quad g_0^{ij} = \text{const.}, \tag{1.2}$$

$$[p, q]_k(z) = b_k^{ij}(p_i(z)q_j'(z) - q_i(z)p_j'(z)), \quad b_k^{ij} + b_k^{ji} = C_k^{ij} = \frac{\partial g^{ij}}{\partial u^k}. \tag{1.3}$$

From the Jacobi identity, the tensor b_k^{ij} by Eq. (1.3) defines a local translationally invariant Lie algebra of first order if and only if $\{b_k^{ji}\}$ is the set of structure constants of a new finite-dimensional algebra A with a bilinear product $(x, \ y) \to xy$ satisfying

$$e_i e_j = \sum_{k=1}^{n} b_k^{ji} e_k, \tag{1.4}$$

$$(x, y, z) = (y, x, z), \tag{1.5}$$

$$(xy)z = (xz)y \tag{1.6}$$

for any $x, y, z \in A$. Here $\{e_1, e_2, \ldots, e_n\}$ is a basis of A and $(x, y, z) = (xy)z - x(yz)$. (Note that we use the left-symmetry here, where the right-symmetry was used in Refs. [1–4].)

The algebra A satisfying Eqs. (1.5) and (1.6) is called a "Novikov algebra" by Osborn et al. [5–10]. It also has a close connection to some Hamiltonian operators in the formal variational calculus [11–14] and some non-linear partial differential equations, such as KdV equations [1,11,12]. On the other hand, Novikov algebras are a special class of left-symmetric algebras which only satisfy Eq. (1.5). Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [15–18]. In fact, let G be a Lie group with a left-invariant affine structure, then this structure induces a flat torsion free left-invariant affine connection ∇ on G, that is, a connection in the tangent bundle $T(G) = \mathcal{G}$ with zero torsion and zero curvature

$$\nabla_x y - \nabla_y x - [x, y] = 0 \quad \text{(zero torsion)}, \tag{1.7}$$

$$\nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z = 0 \quad \text{(zero curvature)}, \tag{1.8}$$

where $x, y, z \in \mathcal{G}$ are arbitrary left-invariant vector fields. Then the bilinear product on the Lie algebra $\mathcal{G}$ of G by $xy = \nabla_x y$ is a left-symmetric algebra. Such a connection was also discussed in Ref. [1].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx, \tag{1.9}$$

defines a sub-adjacent Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote the left and right multiplications, respectively, i.e., $L_x(y) = xy$, $R_x(y) = yx \, \forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. If every R_x is nilpotent, then A is called right-nilpotent or transitive. The transitivity corresponds to the completeness of the affine connections [15,16].

The study of Novikov algebras is obviously difficult due to the non-associativity. Although for the finite-dimensional Novikov algebras, there has been some progress [4,19–21], there are still many open questions. In particular, only the classification of Novikov algebras in low dimensions is known and there are very little detailed examples in higher dimensions. Such a situation seriously hinders the further development of Novikov algebras. One of the ways to find more examples is to find some special Novikov algebras, that is, some Novikov algebras satisfying certain additional conditions.

On the other hand, Novikov algebras have the close relations with geometry, in particular, it is quite important to use the theory of Lie groups. However, the relationship between the Novikov algebras and their sub-adjacent Lie algebras cannot ensure any consequences for the group structure. To solve this problem, we need to consider the more intrinsic connections defined by Novikov algebra structures, that is, the connections which are adapted to the automorphism structure of a Lie group. The study of such a structure was begun in Ref. [22] for a general left-symmetric algebra. Just like in the introduction in Ref. [22], this can be considered as a first approach to the problem of finding the Lie groups which admit complete, locally flat (zero curvature and torsion), left-invariant connections. The structure is given as follows: let G be a Lie group with Lie algebra $\mathcal{G}$, and $\mathrm{Aut}(\mathcal{G})$ is the group of automorphisms of Lie algebra $\mathcal{G}$. The local automorphism structure of G is the principal fiber bundle of frames of G obtained by the extension to $\mathrm{Aut}(\mathcal{G})$ of a left-invariant parallelism of G. Its fibers are unique to a right translation in G's frame bundle $R(G)$.

In this paper, we commence a study of the Novikov algebra structures adapted to the structures defined above and as a consequence we can see that most of left-symmetric algebras obtained in Ref. [22] are Novikov algebras. This paper is organized as follows. In Section 2, we discuss the algebraic properties of Novikov algebra structures adapted to the automorphism structure of a Lie group. In Section 3, we give the classification of these structures in low dimensions. In Section 4, we obtain some examples in higher dimensions. In Section 5, we give some conclusions based on the discussion in the previous sections.

2. Novikov derivation algebras

From the discussion in Ref. [22], we can know that a left-invariant connection ∇ on G is adapted to the automorphism structure of G if and only if the linear mapping $\theta : \mathcal{G} \to \mathcal{H}(\mathcal{G})$ defined by $\theta(x) = \nabla_x$ takes values in the algebra $\mathrm{Der}(\mathcal{G})$, where $\mathrm{Der}(\mathcal{G})$ is the Lie algebra of the derivations of the Lie algebra $\mathcal{G}$. Hence, we call a Novikov algebra A is a derivation algebra if its left multiplications L_x or its right multiplications R_x are derivations of Lie algebra $\mathcal{G}(A)$. Therefore the Lie group G possesses a left-invariant locally flat connection defined by a Novikov algebra which is adapted to the structure of its automorphisms if and only if the Lie algebra $\mathcal{G}$ is sub-adjacent to a Novikov derivation algebra. Furthermore, we have the following claim.

Claim. Let A be a Novikov algebra. Then A is a derivation algebra if and only if the left multiplication operators are commutative.

In fact, for any $x, y, z \in A$, we have

$$\begin{aligned} L_x([y,z]) &= [L_x(y), z] + [y, L_x(z)] \Leftrightarrow x(yz - zy) \\ &= xy(z) - z(xy) + y(xz) - (xz)y \Leftrightarrow x(yz) - (xy)z - y(xz) \\ &= x(zy) - (xz)y - z(xy) \Leftrightarrow (yx)z = (zx)y \Leftrightarrow L_{yx} = R_x R_y. \end{aligned}$$

Hence we can obtain

$$[L_x, L_y] = L_{[x,y]} = L_{xy} - L_{yx} = R_y R_x - R_x R_y = 0.$$

Corollary 2.1. *A Novikov algebra is a derivation algebra if and only if for any x in the derived Lie ideal $[\mathcal{G}(A), \mathcal{G}(A)]$, we have $L_x = 0$.*

Obviously, all commutative Novikov algebras (they are associative and commutative) are derivation algebras. So we mainly study the non-commutative finite-dimensional Novikov derivation algebras in this paper. In fact, we have the following structure theorem [22]. Let A be a finite-dimensional Novikov derivation algebra. Then A has a unique decomposition as a direct sum of two ideals

$$A = A_0 \oplus A_1, \tag{2.1}$$

where A_0 is a transitive Novikov algebra and A_1 is an algebra with an identity, and

$$A_0 \supset N(A) \supset [\mathcal{G}(A), \mathcal{G}(A)], \qquad A_1 \subset C(A). \tag{2.2}$$

Here $N(A) = \{x \in A | L_x = 0\}$, $C(A) = \{x \in A | L_x = R_x\} = \{x \in \mathcal{G}(A) | [x, y] = 0\, \forall y \in \mathcal{G}(A)\}$. It is easy to see that both $N(A)$ and $C(A)$ are ideals of A and $C(A)$ is the center of Lie algebra $\mathcal{G}(A)$.

Corollary 2.2. *Let A be a non-commutative Novikov derivation algebra. If the center $C(A)$ of its sub-adjacent Lie algebra is zero or $C(A) \subset [\mathcal{G}(A), \mathcal{G}(A)]$, then A is transitive. In particular, the Novikov derivation algebras on Heisenberg Lie algebras must be transitive.*

3. The classification of Novikov derivation algebras in low dimensions

In Refs. [19,21], we have obtained the classification of Novikov algebras in dimension ≤ 3 and the transitive Novikov algebras on four-dimensional nilpotent Lie algebras. Through Corollary 2.1, we can obtain the following classification results (over the complex number field). Recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n c_{11}^k e_k & \cdots & \sum_{k=1}^n c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^n c_{n1}^k e_k & \cdots & \sum_{k=1}^n c_{nn}^k e_k \end{pmatrix}, \tag{3.1}$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^n c_{ij}^k e_k$.

Two-dimensional Novikov derivation algebras:

$$\text{Commutative}: \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix},$$

$$\text{Non-commutative}: \quad \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}.$$

Three-dimensional Novikov derivation algebras:

$$\text{Commutative}: \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}, \quad \begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \quad \begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix},$$

$$\text{Non-commutative}: \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & e_2 \end{pmatrix}, \quad l \neq 1, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}, \quad |l| \leq 1, \quad l \neq 0, \quad \begin{pmatrix} 0 & 0 & 0 \\ -e_1 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}.$$

Four-dimensional transitive Novikov derivation algebras on nilpotent Lie algebras:

$$\text{Commutative}: \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \\ 0 & e_1 & 0 & e_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_2 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_2 & -e_1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Non-commutative :

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & e_1 & 0 \\ 0 & -e_1 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & 0 \\ 0 & -e_1 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & e_1 & 0 \\ 0 & -e_1 & te_1 & 0 \\ 0 & 0 & 0 & e_1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_1 & e_1 & 0 \\ 0 & 0 & te_1 & 0 \\ 0 & 0 & 0 & -e_1 \end{pmatrix}, \quad t \geq 0, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & e_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & te_1 \\ 0 & 0 & e_1 & 0 \\ 0 & e_1 & 0 & e_2 \end{pmatrix}, \quad t \neq 1, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & te_1 \\ 0 & 0 & 0 & e_1 \\ 0 & e_1 & 0 & e_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_1 & e_2 \\ 0 & e_1 & e_1+e_2 & e_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2 \\ 0 & 0 & -e_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \\ 0 & 0 & -e_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & -e_2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 \\ 0 & 0 & -e_2 & e_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & te_1 \\ 0 & 0 & -te_1 & e_1 \end{pmatrix}, \quad t > 0,$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & e_2 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & te_1 & e_3 \end{pmatrix}, \quad t \neq 1,$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & (1+t)e_2 \\ 0 & 0 & (1-t)e_2 & -e_1 \end{pmatrix}, \quad t > 0, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1+te_2 \\ 0 & 0 & -e_1-te_2 & e_2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_1+e_2 \\ 0 & 0 & -e_1+e_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & (1+t)e_2 \\ 0 & 0 & (1-t)e_2 & 0 \end{pmatrix}, \quad t > 0,$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 \\ 0 & e_1 & e_2 & e_3 \end{pmatrix}.$$

4. Some Novikov derivation algebras in higher dimensions

It is quite interesting to see that most of left-symmetric derivation algebras given in Ref. [22] are Novikov algebras. Thus, in this section, we can obtain some Novikov derivation algebras in higher dimensions. For self-contained, we give these examples (Examples 4.1–4.5) a brief description. First of all, we give some examples of Novikov derivation algebras in dimension 5.

Example 4.1. There are two important Novikov derivation algebras in dimension 5 given in Ref. [22] with the following characteristic matrices, respectively:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -e_1 & -e_2 & 0 & e_1 & e_1 \\ -e_1 & -e_1 & e_1 - e_2 & \frac{1}{2}(e_1 + e_2) & \frac{1}{2}(e_1 + e_2) \\ -e_1 & -e_1 & e_1 - e_2 & \frac{1}{2}(e_1 + e_2) & \frac{1}{2}(e_1 + e_2) \end{pmatrix},$$

$$\begin{pmatrix} \lambda e_3 + \beta e_4 + \mu e_5 & e_3 + \lambda e_4 + \gamma e_5 & e_4 & e_5 & 0 \\ \lambda e_4 + \gamma e_5 & e_4 + \delta e_5 & e_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The sub-adjacent Lie algebra of the former is

$$[e_1, e_3] = [e_1, e_4] = [e_1, e_5] = [e_2, e_3] = [e_2, e_4] = [e_2, e_5] = e_1,$$
$$[e_1, e_2] = [e_4, e_5] = 0, \qquad [e_3, e_4] = [e_3, e_5] = e_2.$$

The sub-adjacent Lie algebra of the latter is (non-zero products)

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4, \qquad [e_1, e_4] = e_5, \qquad [e_2, e_3] = e_5.$$

Example 4.2. We can construct a series of Novikov derivation algebras in dimension ≥ 5 through the extension of a five-dimensional Novikov derivation algebra. Let A be the Lie algebra in dimension 5 with the following non-zero products:

$$[e_1, e_3] = e_5, \qquad [e_1, e_3] = e_3, \qquad [e_2, e_4] = e_4.$$

A Novikov derivation product on A is obtained by taking for the left multiplications the following endomorphisms:

$$L_{e_1} = \mathrm{ad}(e_1), \qquad L_{e_2} = \mathrm{ad}^2(e_2), \qquad L_{e_3} = L_{e_4} = L_{e_5} = 0,$$

where ad is the adjoint operator of Lie algebra, that is, $\mathrm{ad}(x)(y) = [x, y]$. Consider the Lie algebra $A' = A \times \mathbf{C}e_6$ obtained from A by imposing

$$[e_1, e_5] = e_6, \qquad [e_i, e_6] = 0 \quad \text{for } 1 \leq i \leq 6.$$

The Novikov derivation product on A' is given as

$$L'_{e_1} = \mathrm{ad}'(e_1), \qquad L'_{e_2} = \mathrm{ad}'^2(e_2), \qquad L'_{e_3} = L'_{e_4} = L'_{e_5} = L'_{e_6} = 0.$$

Thus, by a series of such extensions we can obtain a series of Novikov derivation algebras.

We can see that in the above examples, the sub-adjacent Lie algebras are 2-solvable Lie algebras, that is, the derived ideal $[\mathcal{G}(A), \mathcal{G}(A)]$ is Abelian. We can see that such a situation is not accident. In fact, we have no any example of a Lie algebra with an order of solvability greater than 2 which is sub-adjacent to a Novikov derivation algebra. Furthermore, we have the following example.

Example 4.3. Let A be a 2-solvable Lie algebra. Suppose A can be decomposed as a direct sum of sub-spaces $A = \mathcal{D}(A) \oplus S$ with $[S, S] \subset C(A)$, where $\mathcal{D}(A) = [A, A]$ and $C(A)$ is the center of A. For every element a in A, we denote by $a_{\mathcal{D}}$ and a_S the respective components of a in $\mathcal{D}(A)$ and S. Then

$$ab = [a_S, b_{\mathcal{D}} + \tfrac{1}{2} b_S]$$

defines a Novikov derivation product on A.

Example 4.4. There exists a Novikov derivation product on any 2-solvable Lie algebra with trivial center. In fact, from the discussion in Ref. [22], such a Lie algebra A has a decomposition

$$A = \mathcal{D}(A) \oplus C,$$

where C is an Abelian Cartan subalgebra of A. Then A satisfies the condition in Example 4.3 since $[C, C] = \{0\} = C(A)$. Thus the Novikov derivation product on A can be defined by

$$L_{a_{\mathcal{D}}} = 0, \qquad L_{a_C} = \mathrm{ad}(a_C),$$

where $a_{\mathcal{D}} \in \mathcal{D}$, $a_C \in C$.

Example 4.5. In fact, there are certain kinds of 2-solvable Lie algebras with the trivial center having the property that it is sub-adjacent to a unique Novikov derivation structure. Such an example can be obtained from Ref. [22]. Let A be an n-dimensional Lie algebra with the product

$$[e_i, e_j] = 0, \quad i, j \geq 2, \qquad [e_1, e_i] = \lambda_i e_i,$$
$$i \geq 2, \quad \lambda_i \neq 0, \text{ the } \lambda_i \text{ being pairwise distinct.}$$

The (unique) Novikov derivation structure is given by

$$e_1 e_1 = 0, \qquad e_1 e_i = \lambda_i e_i, \qquad e_i e_j = 0, \quad i, j \geq 2.$$

At the end of this section, we give an example of Novikov derivation algebra on a filiform Lie algebra with Abelian commutator subalgebra. A filiform Lie algebra A in dimension n is a $(n-1)$-step nilpotent Lie algebra, that is, the lower central series $\{A^k\}$ of A ($A^0 = A$ and $A^k = [A^{k-1}, A]$ for $k \geq 1$) satisfying $A^{n-1} = 0$, $A^{n-2} \neq 0$. The study of filiform Lie algebra is quite important [23]. For example, the first example of the nilpotent Lie algebra which is not sub-adjacent to a left-symmetric algebra is a filiform Lie algebra [24].

Example 4.6. Let A be a filiform Lie algebra with Abelian commutator subalgebra. Then the product is given by Burde [23] (non-zero products)

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, n-1,$$
$$[e_2, e_i] = \sum_{k=i+2}^{n} \alpha_{2,k-i+3} e_k, \quad i = 3, \ldots, n-2$$

with parameters $\alpha_{2,s}$, where $5 \le s \le n$. Then it is easy to check that the algebra given by the following products is a Novikov derivation algebra:

$$e_1 e_i = e_{i+1}, \quad i = 2, \ldots, n-1, \qquad e_2 e_i = [e_2, e_i], \quad i = 3, \ldots, n-2,$$
$$e_2 e_2 = \alpha_{2,5} e_4 + \cdots + \alpha_{2,n} e_{n-1}, \qquad e_i e_j = 0 \quad \text{otherwise}.$$

5. Conclusions and discussion

From the discussion in the previous sections, we have seen the importance of the study of Novikov derivation algebras. Moreover, we have the following conclusions:

(1) Comparing with the results in Refs. [19–21], we can see that every transitive Novikov algebra in dimension ≤ 4 on 2-step nilpotent Lie algebra (the derived ideal is in the center of Lie algebra) is a derivation algebra.

(2) Except the type

$$\begin{pmatrix} 0 & 0 & 0 \\ -e_1 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

which is the direct sum of the (unique) two-dimensional non-commutative transitive Novikov algebra and the field, every non-commutative Novikov derivation algebra in dimension ≤ 3 is transitive.

(3) We would like to point out that the structure theorem given in Section 2 is not the same as the fundamental structure theory of Novikov algebras given by Zel'manov [4]. Zel'manov proved that a finite-dimensional Novikov algebra A over an algebraically closed field with characteristic 0 contains a (unique) largest transitive ideal $R(A)$ (is called the radical of A) and the quotient algebra $A/R(A)$ is a direct sum of fields. Obviously, $A_0 \subset R(A)$. However, A_0 does not necessarily equal to $R(A)$, in particular, in the case of commutative Novikov algebras. This means that every Novikov derivation algebra is not necessarily the direct sum of $R(A)$ and fields. Despite this, it is interesting to see that for the non-commutative Novikov derivation algebras in dimension ≤ 3, these two structure theorems coincide (see Conclusion (2)).

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